Intermittency inhibited by transport: An exactly solvable model

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Transport is incorporated in a discrete-time stochastic model of a system undergoing autocatalytic reactions of the type $A \to 2A$ and $A \to 0$, whose population field is known to exhibit spatiotemporal intermittency. The temporal evolution is exactly solved, and it is shown that if the transport process is strong enough, intermittency is inhibited. This inhibition is nonuniform, in the sense that, as transport is strengthened, low-order population moments are affected before the high-order ones. Numerical simulations are presented to support the analytical results.

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I. INTRODUCTION

Spatiotemporal intermittency is a phenomenon underlying the behavior of a wide class of stochastic systems, ranging from population dynamics to turbulent fluids. In intermittent systems the relevant fields are different from zero in very localized regions of space and time, whose evolution governs in consequence the behavior of the whole system.

In fully developed viscous turbulence, for instance, the vorticity field—as well as the enstrophy, i.e., the rate of energy dissipation [1]—is concentrated along the vortex lines [2], where it takes "quasisingular" values [3]. In turbulent plasmas, the magnetic field becomes trapped in vortices and, therefore, exhibits intermittency [4].

In population dynamics, spatial and temporal fluctuations of birth and death rates determine a strongly nonhomogeneous population distribution [5]. It is characterized by sharp, very scattered spikes, produced by the accumulation of favorable birth events. These population peaks can be annihilated by death, at the same time that others are forming elsewhere. Although these peaks correspond to very rare events, they dominate the evolution of the system, as the main part of the total population concentrates there.

The role of intermittency in fluctuating population dynamics suggests that it can also be relevant to the kinetics of autocatalytic chemical reactions. At the opposite length-scale extreme, this phenomenon has also been detected in the matter distribution of the Universe [6].

An important question about the evolution of intermittent systems—associated to the possibility of controlling the effect of accumulation of stochastic events—is whether the development of intermittency can be inhibited by a dissipative process, such as energy loss in the case of turbulence, or diffusion in population dynamics. This latter case was treated in detail in Ref. [7]. There, it was shown that a population whose density $n(\mathbf{r},t)$ is governed by the reaction-diffusion equation

$$\partial_t n - D\ \nabla_{\mathbf{r}}^2 n = f(\mathbf{r},t)\ n, \tag{1}$$

where $f(\mathbf{r},t)$ is a stochastic (Gaussian) process, develops the high spikes characteristic of intermittency for small D. On the other hand, when diffusivity is very large, the occurrence of intermittency depends on the dimensionality of the system. For one-dimensional populations, intermittency persists even in the limit $D \to \infty$. In higher-dimensional systems, instead, this phenomenon is partially limited—but not completely suppressed—by diffusion.

The exactly solvable model presented in this paper describes an evolving population subject to a transport process which can be assimilated to diffusion in an high dimensional space. When this process is sufficiently strong intermittency is completely inhibited, but for intermediate strengths, a partial inhibition of the same type observed for lower-dimensional diffusion occurs. Numerical simulations validate the exact solution of the model, and show that, eventually, fluctuations can strongly affect the average population dynamics.

II. THE MODEL AND ITS SOLUTION

Consider a system of particles evolving on an N-site lattice. Initially, there is exactly one particle at each site, so that the total population equals N. Now, at each time step t, the population of each site $n(x,t)$ is exactly doubled or completely annihilated, with probability 1/2 for each process:

$$n(x,t+1) = \mathcal{R}[n(x,t)] = \begin{cases} 2n(x,t) & \text{with probability } 1/2, \\ 0 & \text{with probability } 1/2. \end{cases} \tag{2}$$

Here, $\mathcal{R}$ is an operator describing birth and death events. This model, originally proposed by Zeldovich [5] as a paradigm of intermittency, represents also a chemical system undergoing the autocatalytic reactions

$$\begin{aligned} A &\longrightarrow 2A, \\ A &\longrightarrow 0. \end{aligned} \tag{3}$$

It is easy to see that, after t time steps, the population at a given site x is $n(x,t) = 2^t$ with probability 2^{-t}, or $n(x,t) = 0$, with the complementary probability $1 - 2^{-t}$. Thus, as times elapses, the population concentrates in increasingly rare and high spikes, characteristic of intermittency.

This strongly non-Gaussian distribution can be characterized by its moments. The mth order moment of the population field is defined as

$$\langle n^m \rangle = \frac{1}{N} \sum_x n(x,t)^m, \tag{4}$$

and its average over realizations is therefore

$$\langle n^m \rangle = 2^{t(m-1)} \ \ (m \geq 1). \tag{5}$$

For $m = 1$, the mean particle number per site $\langle n \rangle$ equals unity, and is independent of time. This indicates that the average total population remains constant, the system being in the threshold of a population explosion [8]. Higher order moments, instead, grow exponentially with time, and their growth rate increases with the moment order. This particular time dependence of the population moments has been identified as a typical feature of intermittency [5].

Consider now the following transport process. At each time step, every particle in the system abandons its site with probability α and hops to a different site, chosen at random with homogeneous probability. Thus, in the average, a fraction α of the population at each site leaves its position and spreads uniformly over the remainder of the lattice. Consequently, a system subject to this sole process would evolve towards a spatially homogeneous state.

This transport process can be associated to diffusion on a lattice with high connection number, for instance, in a high-dimensional space. On a low-dimensional lattice, it is stronger—i.e., more effective—than diffusion. In fact, considering for instance an initial condition in which the population is concentrated on a single site, it can be seen that, at small times, the mean square displacement grows as t^2. At large times, of course, the finite size of the system implies that the population distribution becomes homogeneous.

Averaging over realizations, the transport process can be represented by an operator $\mathcal{T}$ such that

$$n(x,t+1) = \mathcal{T}[n(x,t)] = (1-\alpha)n(x,t) + \alpha\langle n \rangle. \tag{6}$$

Here, $\langle n \rangle$ is the mean particle number per site. As well as $\mathcal{R}$, the operator $\mathcal{T}$ preserves the total number of particles, so that $\langle n \rangle$ is a constant. According to the initial condition considered for the reaction model described above, $\langle n \rangle$ will be hereafter put equal to unity.

The combination of reaction and transport can be carried out in two ways, according to the order in which these processes are applied at each time step. If transport acts first, then reaction, their combined effect is described by the operator $\mathcal{RT}$, which produces

$$n(t+1) = \mathcal{RT}[n(t)] = \begin{cases} \beta n(t) + 2\alpha & \text{with probability } 1/2, \\ 0 & \text{with probability } 1/2, \end{cases} \tag{7}$$

with $\beta = 2(1-\alpha)$. For simplicity in notation, the spatial variable has been omitted, the whole process being local in space. The following analysis will concentrate in this form of combining transport and reaction. The alternative way—described by the operator $\mathcal{TR}$—does not present qualitative differences in the results. The evolution of the population at any site can be found by solving the iterative linear stochastic equation (7). Taking into account that $n(0) = 1$, this gives

$$n(t) = \begin{cases} \beta^t + \frac{2\alpha}{2\alpha-1}[1-\beta^t] & \text{with probability } 1/2^t, \\ \frac{2\alpha}{2\alpha-1}[1-\beta^k] & \text{with probability } 1/2^{k+1}, 0 \leq k < t. \end{cases} \tag{8}$$

Here, the first line corresponds to the situation in which no annihilation has occurred during the whole t-step evolution. The second line represents the case in which the last annihilation event has occurred k time steps ago.

The highest among these $t+1$ possible values for the site population at time t, $n_{\max}(t)$, is given by the first line in (8). For $\alpha = 0$, this is the only nonvanishing value for $n(t)$, and equals $n_{\max}(t) = 2^t$. This reproduces the evolution of the Zeldovich model, Eq. (2). For $\alpha \neq 0$, the large-time behavior is determined by the value of $\beta = 2(1-\alpha)$. If $\beta > 1$ ($\alpha < 1/2$), the exponential growth persists, although the growth rate decreases as α varies from 0 to 1/2. Asymptotically, these highest population spikes behave as

$$n_{\max}(t) \approx \frac{1}{1-2\alpha} \beta^t. \tag{9}$$

For $\beta < 1$ ($\alpha > 1/2$), instead, $n_{\max}$ approaches a constant level

$$n_{\max}(t) \longrightarrow \frac{2\alpha}{2\alpha-1}, \tag{10}$$

and this value decreases as α grows.

These results show clearly that the exponential growth of the population per site can disappear if transport is strong enough, more precisely, if the jump probability α is greater than 1/2. This suggests that intermittency is also inhibited for $\alpha > 1/2$, as the evolution of the population moments should reflect this limitation in the population growth. The details in this process of intermittency inhibition are studied in the next section.

III. EVOLUTION OF THE POPULATION MOMENTS

As said before, the temporal behavior of the population moments defined in (4) characterizes intermittency.

In fact, the moments of a population distribution with high and very scattered spikes increase as their order grows. In an intermittent system with a more or less homogeneous initial distribution, this trend is developed as time elapses. If, on the contrary, high-order moments remain relatively small, one can assert that the system does not exhibit intermittency.

According to the results presented in Sec. II, the reaction-transport model defined by Eq. (7) should be intermittent for small values of α and become regular when α reaches a certain critical value. In fact, it results that the population moments exhibit a transition of the type observed for $n(t)$. However, the rather complicated form of the exact solution (8) makes it difficult to write down the detailed expression for the average moments $\langle n^m \rangle$.

It is instead relatively simple to give the asymptotic large-time behavior of these moments. Assuming that the mth order moment diverges for large times, the dominant contribution is given by the first line in (8), and one gets

$$\langle n^m \rangle \approx \left[\left(\frac{1}{1-2\alpha} \right)^m - \frac{1}{2-\beta^m} \left(\frac{2\alpha}{1-2\alpha} \right)^m \right] \left(\frac{\beta^m}{2} \right)^t . \qquad (11)$$

In this expression, the temporal evolution affects only the last factor. According to this, one sees that $\langle n^m \rangle$ effectively diverges only if $\beta^m/2 > 1$, i.e., if

$$\alpha < 1 - \left(\frac{1}{2} \right)^{\frac{m-1}{m}} = \alpha_m . \qquad (12)$$

The critical value α_m, which depends on the moment order, establishes the upper bound for the exponential divergence of $\langle n^m \rangle$. It increases with m from $\alpha_2 \approx 0.29$, and approaches $1/2$ as $m \to \infty$.

This result indicates that the inhibition in the moment growth is not uniform with respect to the moment order. The transport process is able to control first the divergence of low-order moments, and has to be strenghtened to prevent the higher-order ones from increasing indefinitely. Thus, according to the large-time behavior of the population moments, three intervals for the jump probability α can be distinguished. For $\alpha < \alpha_2$, the population is absolutely intermittent, in the sense that all its moments $\langle n^m \rangle$ ($m > 1$) diverge exponentially.

For intermediate values of α, $\alpha_2 < \alpha < 1/2$, the population at large times is characterized by a set of low-order finite moments, whereas the other ones diverge. One can say that intermittency has been partially inhibited. The limiting value of the finite moments can be calculated from (8) as the contribution of the time-independent part:

$$\langle n^m \rangle \longrightarrow \left(\frac{2\alpha}{2\alpha - 1} \right)^m \sum_{r=0}^{m} \frac{m!}{r!(m-r)!} \frac{(-1)^r}{2-\beta^r} . \qquad (13)$$

Finally, for $\alpha > 1/2$, the exponential growth of all the population moments is prevented, in accordance with the fact that the population at all sites remains finite as time elapses. Intermittency has been completely inhibited.

IV. NUMERICAL RESULTS AND DISCUSSION

Numerical simulations make it possible to compare the theoretical results presented above—which correspond to averages over realizations—with the actual evolution of the system. These simulations have been performed along the lines described in Ref. [9].

Since the average evolution is local in space, the relevant parameter defining the size N_S of the numerical simulations is the product of the number of lattice sites N times the number of realizations N_R, $N_S = N_R N$. The results presented here correspond to N_S ranging from 10^7 to 10^8, for various values of the jump probability α.

Figure 1 shows the evolution of the population moments $\langle n^m \rangle$ for $m = 1$ to 4, with $\alpha = 0.1$. This value of the jump probability corresponds to completely developed intermittency, so that the moments are expected to grow exponentially as time elapses. In the figure, full lines correspond to the numerical simulations, whereas dashed lines represent the theoretical prediction (11). Both results show a good agreement up to a certain critical time $t_c \approx 23$. From then on, fluctuations play a relevant role in the evolution of the numerical system, and the actual evolution differs appreciably from the theoretical result. The critical time t_c grows as the simulation size N_S is increased, so that the average results are expected to correctly describe the evolution over the whole time range only for an infinite system, $N \to \infty$.

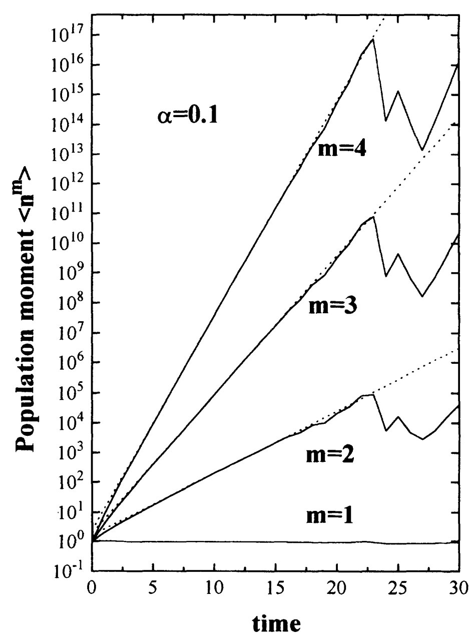

FIG. 1. Population moments from first to fourth order, for a jump probability $\alpha = 0.1$ (intermittency developed). The initial population is homogeneous, with one particle per site. Full lines represent the results of numerical simulations for $N_S = 10^7$ on a 10^4-site lattice, and dotted lines stand for the corresponding theoretical results.

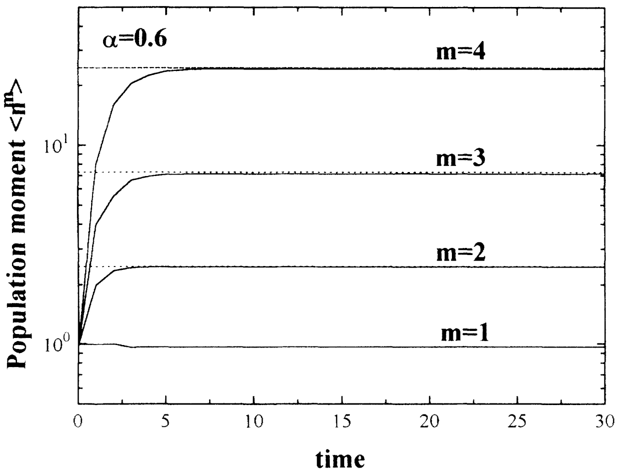

FIG. 2. As in Fig. 1, for $\alpha = 0.6$ (intermittency inhibited).

On the other hand, when intermittency is absolutely inhibited, the theoretical prediction (13) gives a good description for arbitrarily long times, and the effect of fluctuations seems to be negligible. This can be clearly seen in Fig. 2, which displays numerical and theoretical results for $\alpha = 0.6$. Observe that, after four or five time steps, all the moments have practically reached their asymptotic value.

In the transition zone, $\alpha_2 < \alpha < 1/2$, the system proves to be particularly susceptible to fluctuations. Simulations performed for $\alpha = 0.35$, shown in Fig. 3, required to take $N_S = 10^8$—one order of magnitude higher than for Figs. 1 and 2. For this value of the jump probability, $\langle n^2 \rangle$ should asymptotically approach a finite value, whereas all the higher order moments should diverge exponentially. In spite of this relatively high simulation size the critical time for the fluctuations to become relevant is $t_c \approx 17$. After this time, both $\langle n^3 \rangle$ and $\langle n^4 \rangle$ show a strongly irregular behavior. Meanwhile, the second-order moment grows very slowly towards the asymptotic value predicted by Eq. (13).

The following features in the behavior of the exactly solvable model analyzed here deserve to be pointed out.
• The transport process—which can be seen as a version of diffusion on a many-dimensional lattice with very high connectivity—is able to completely inhibit the development of intermittency induced by birth and death (or autocatalytic chemical) events, if it is sufficiently strong. Since it has already been proved that diffusion cannot eliminate intermittency in less than four dimensions [7], this result suggests the existence of a critical spatial dimension d_c. Such inhibition should occur in systems with $d > d_c$. This critical dimension could be infinite.
• When transport is weak, the maximum particle number per site grows exponentially with time. As transport is strengthened, at a certain point this growth is suddenly suppressed. However, the process of intermittency inhibition begins before the occurrence of this sudden transition and proceeds gradually, limiting first the growth of low-order population moments. At the critical point in which the large-time divergence of the maximum population is suppressed, the growth of all moments is also

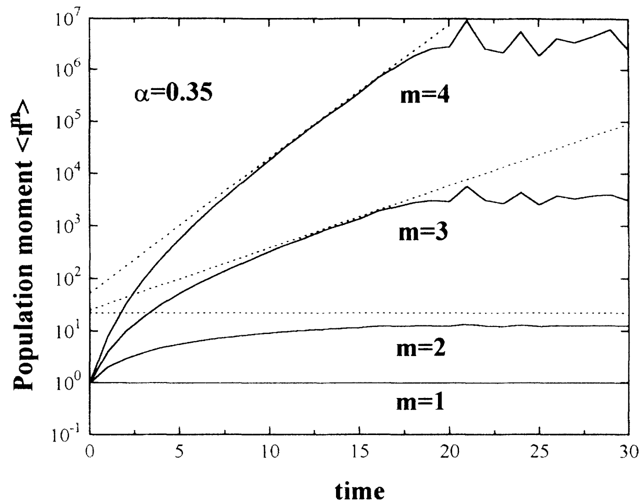

FIG. 3. As in Fig. 1, for $\alpha = 0.35$ (transition zone). These simulations correspond to $N_S = 10^8$.

limited. This gradual inhibition has been also detected in finite-dimensional diffusion. In the limit of large diffusivity, in fact, low-order moments of three-dimensional populations evolve towards finite values, although intermittency cannot be definitively eliminated [7].
• In this model, the development of high spikes in the population distribution is due to the accumulation of favorable birth events. Thus the system evolution is essentially driven by fluctuations. For a system of a given size, the effects of stochastic events are expected to prevail after a certain time, during which they accumulate to become relevant. In this transient, the system is well described by the average results obtained here. This fact is apparent in the numerical simulations, when transport is not strong enough to inhibit intermittency. An appropriate description of the regime in which fluctuations prevail, instead, would require a more detailed analysis of their statistical properties. On the other hand, if intermittency has been suppressed, fluctuations do not play a relevant role and the average results are correct at all times. This would be also the case—even in the intermittent regime—for an infinite system, where fluctuations are naturally negligible.

Although the outlined conclusions apply to the specific model presented here, they may be characteristic of the interplay between certain transport processes and the development of intermittency. Therefore, the model could serve as a paradigm of many fluctuation-driven systems found in actual applications—such as in population dynamics [7] and in plasma physics [10]—and become particularly relevant when the accumulation of fluctuations leads to catastrophic effects [11].

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